Problem 1.24

Derive the three quotient rules.

Solution

Proof of Quotient Rule 1

The aim is to prove that

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}. \tag{1}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\frac{f}{g}\right) \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\left(\frac{f}{g}\right) \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\frac{\partial f}{\partial x_i}g - \frac{\partial g}{\partial x_i}f}{g^2} \\
&= \frac{\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial f}{\partial x_i}g - \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial g}{\partial x_i}f}{g^2} \\
&= \frac{g\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial f}{\partial x_i}\right) - f\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial g}{\partial x_i}\right)}{g^2} \\
&= \frac{g\nabla f - f\nabla g}{g^2}
\end{aligned}$$

Proof of Quotient Rule 2

The aim is to prove that

$$\nabla \cdot \left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla \cdot \mathbf{A}) - \mathbf{A} \cdot (\nabla g)}{g^2}. \tag{2}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \cdot \left(\frac{\mathbf{A}}{g}\right) = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \frac{1}{g}\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\left(\frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial}{\partial x_i}\left(\frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{A_i}{g}\right) \\
&= \sum_{i=1}^{3} \frac{\frac{\partial A_i}{\partial x_i} g - \frac{\partial g}{\partial x_i} A_i}{g^2} \\
&= \frac{\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} g - \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} A_i}{g^2} \\
&= \frac{g\left(\sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i}\right) - \sum_{i=1}^{3} A_i \frac{\partial g}{\partial x_i}}{g^2} \\
&= \frac{g\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial A_j}{\partial x_i}\right) - \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} A_i \frac{\partial g}{\partial x_j}}{g^2} \\
&= \frac{g\left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i}\right] - \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_i \frac{\partial g}{\partial x_j}}{g^2} \\
&= \frac{g\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] - \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial g}{\partial x_j}\right)}{g^2} \\
&= \frac{g(\nabla \cdot \mathbf{A}) - \mathbf{A} \cdot (\nabla g)}{g^2}
\end{aligned}$$

Proof of Quotient Rule 3

The aim is to prove that

$$\nabla \times \left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla g)}{g^2}. \tag{3}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \times \left(\frac{\mathbf{A}}{g}\right) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \frac{1}{g}\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\left(\frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(\frac{A_j}{g}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\frac{\partial A_j}{\partial x_i} g - \frac{\partial g}{\partial x_i} A_j}{g^2} \\
&= \frac{\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} g - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial g}{\partial x_i} A_j}{g^2} \\
&= \frac{g\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i}\right) - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k (-\varepsilon_{jik}) A_j \frac{\partial g}{\partial x_i}}{g^2} \\
&= \frac{g\left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i}\right] + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_i) A_j \frac{\partial g}{\partial x_i}}{g^2} \\
&= \frac{g\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] + \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial g}{\partial x_i}\right)}{g^2} \\
&= \frac{g(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla g)}{g^2}
\end{aligned}$$